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OPERATIONS RESEARCH

Transportation Problem: A Special Case for Linear Programming Problems

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A key problem managers face is how to allocate scarce resources among various activities or projects. Linear programming, or LP, is a method of allocating resources in an optimal way. It is one of the most widely used operations research tools and has been a decision-making aid in almost all manufacturing industries and in financial and service organizations.

In the term *linear programming, programming* refers to mathematical programming. In this context, it refers to a planning process that allocates resources—labor, materials, machines, capital—in the best possible (optimal) way so that costs are minimized or profits are maximized. In LP, these resources are known as *decision variables*. The criterion for selecting the best values of the decision variables (e.g., to maximize profits or minimize costs) is known as the *objective function*. Limitations on resource availability form what is known as a *constraint set*.

The word *linear* indicates that the criterion for selecting the best values of the decision variables can be described by a linear function of these variables; that is, a mathematical function involving only the first powers of the variables with no cross-products. For example, $23X_2$ and $4X_{16}$ are valid decision variables, while $23X_2^2$, $4X_{16}^3$, and $(4X_1 * 2X_1)$ are not.

Operations research (OR) is concerned with scientifically deciding how to best design and operate people–machine systems, usually under conditions requiring the allocation of scarce resources.[1]

This publication, one of a series, is offered to supervisors, lead people, middle managers, and anyone who has responsibility for operations planning in manufacturing facilities or corporate planning over multiple facilities. Practical examples are geared to the wood products industry, although managers and planners in other industries can learn OR techniques through this series.

[1]*Operations Research Society of America*

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We can solve…
relatively large transportation problems by hand.

The entire problem can be expressed in terms of straight lines, planes, or analogous geometrical figures.

In addition to the linear requirements, non-negativity restrictions state that variables cannot assume negative values. That is, it's not possible to have negative resources. Without that condition, it would be mathematically possible to "solve" the problem using more resources than are available.

In earlier reports (see OSU Extension publications list, page 35), we discussed using LP to find optimal solutions for maximization and minimization problems. We also learned we can use sensitivity analysis to tell us more about our solution than just the final optimal solution. In this publication, we discuss a special case of LP, the transportation problem.

The transportation problem

One of the most important and successful applications of quantitative analysis to solving business problems has been in the physical distribution of products, commonly referred to as *transportation problems*. Basically, the purpose is to minimize the cost of shipping goods from one location to another so that the needs of each arrival area are met and every shipping location operates within its capacity. However, quantitative analysis has been used for many problems other than the physical distribution of goods. For example, it has been used to efficiently place employees at certain jobs within an organization. (This application sometimes is called the *assignment problem.*)

We could set up a transportation problem and solve it using the simplex method as with any LP problem (see *Using the Simplex Method to Solve Linear Programming Maximization Problems*, EM 8720, or another of the sources listed on page 35 for information about the simplex method). However, the special structure of the transportation problem allows us to solve it with a faster, more economical algorithm than simplex. Problems of this type, containing thousands of variables and constraints, can be solved in only a few seconds on a computer. In fact, we can solve a relatively large transportation problem by hand.

There are some requirements for placing an LP problem into the transportation problem category. We will discuss those requirements on page 6, after we formulate our problem and solve it using computer software.

Computer solution

First, let's formulate our problem and set it up as a "regular" LP problem that we will solve using the LP software LINDO.*

The XYZ Sawmill Company's CEO asks to see next month's log hauling schedule to his three sawmills. He wants to make sure he keeps a steady, adequate flow of logs to his sawmills to capitalize on the good lumber market. Secondary, but still important to him, is to minimize the cost of transportation. The harvesting group plans to move to three new logging sites. The distance from each site to each sawmill is in Table 1. The average haul cost is $2 per mile for both loaded and empty trucks. The logging supervisor estimated the number of truckloads of logs coming off each harvest site daily. The number of truckloads varies because terrain and cutting patterns are unique for each site. Finally, the sawmill managers have estimated the truckloads of logs their mills need each day. All these estimates are in Table 1.

Table 1.—Supply and demand of sawlogs for the XYZ Sawmill Company.

Logging	Distance to mill (miles)			Maximum truckloads/day
site	Mill A	Mill B	Mill C	per logging site
1	8	15	50	20
2	10	17	20	30
3	30	26	15	45
Mill demand (truckloads/day)	30	35	30	

The next step is to determine costs to haul from each site to each mill (Table 2).

Table 2.—Round-trip transportation costs for XYZ Sawmill Company.

Logging site	Mill A	Mill B	Mill C
1	$ 32*	$ 60	$ 200
2	40	68	80
3	120	104	60

*(8 miles x 2) x ($2 per mile) = $32

We can set the LP problem up as a cost minimization; that is, we want to minimize hauling costs and meet each of the sawmills'

**Solver Suite: LINDO, LINGO, WHAT'S BEST. LINDO Systems Inc., Chicago. 382 pp. This product is mentioned as an illustration only. The Oregon State University Extension Service neither endorses this product nor intends to discriminate against products not mentioned.*

daily demand while not exceeding the maximum number of truckloads from each site. We can formulate the problem as:

Let X_{ij} = Haul costs from Site i to Mill j
i = 1, 2, 3 (logging sites)   j = 1, 2, 3 (sawmills)

Objective function:
MIN $32X_{11} + 40X_{21} + 120X_{31} + 60X_{12} + 68X_{22} + 104X_{32} + 200X_{13} + 80X_{23} + 60X_{33}$

Subject to:

$X_{11} + X_{21} + X_{31} \geq 30$	Truckloads to Mill A
$X_{12} + X_{22} + X_{32} \geq 35$	Truckloads to Mill B
$X_{13} + X_{23} + X_{33} \geq 30$	Truckloads to Mill C
$X_{11} + X_{12} + X_{13} \leq 20$	Truckloads from Site 1
$X_{21} + X_{22} + X_{23} \leq 30$	Truckloads from Site 2
$X_{31} + X_{32} + X_{33} \leq 45$	Truckloads from Site 3

$$X_{11}, X_{21}, X_{31}, X_{12}, X_{22}, X_{32}, X_{13}, X_{23}, X_{33} \geq 0$$

For the computer solution: in the edit box of LINDO, type in the objective function, then “Subject to,” then list the constraints.

Objective function:
MIN $32X_{11} + 40X_{21} + 120X_{31} + 60X_{12} + 68X_{22} + 104X_{32} + 200X_{13} + 80X_{23} + 60X_{33}$

Subject to:
$X_{11} + X_{21} + X_{31} \geq 30$
$X_{12} + X_{22} + X_{32} \geq 35$
$X_{13} + X_{23} + X_{33} \geq 30$
$X_{11} + X_{12} + X_{13} \leq 20$
$X_{21} + X_{22} + X_{23} \leq 30$
$X_{31} + X_{32} + X_{33} \leq 45$

Note, the non-negativity constraint does not have to be typed in because LINDO knows that all LP problems have this constraint.

That’s it! The software will solve the problem. It will add *slack, surplus,* and *artificial variables* when necessary. (For an explanation of slack, surplus, and artificial variables, see an earlier report in this series or consult another of the references on page 35.)

The LINDO (partial) output for the XYZ Sawmill Company transportation problem:

LP OPTIMUM FOUND AT STEP 3
OBJECTIVE FUNCTION VALUE
1) 5760.000

```
VARIABLE        VALUE          REDUCED COST
     X11      20.000000           0.000000
     X21      10.000000           0.000000
     X31       0.000000          44.000000
     X12       0.000000           0.000000
     X22      20.000000           0.000000
     X32      15.000000           0.000000
     X13       0.000000         184.000000
     X23       0.000000          56.000000
     X33      30.000000           0.000000

     ROW   SLACK OR SURPLUS     DUAL PRICES
      2)       0.000000         –76.000000
      3)       0.000000        –104.000000
      4)       0.000000         –60.000000
      5)       0.000000          44.000000
      6)       0.000000          36.000000
      7)       0.000000           0.000000

     NO. ITERATIONS = 3
```

Interpretation of the LINDO output

It took three iterations, or *pivots*, to find the optimal solution of \$5,760. (To solve this small LP by hand would have required computations for at least three simplex tableaus.)

The \$5,760 represents the minimum daily haul costs for the XYZ Sawmill Company from the three logging sites to the three sawmills. We can use the values in the VALUE column to assign values to our variables and determine the log truck haul schedules. For variable X_{11}, which is Site 1 to Mill A, the VALUE—number of truckloads per day—is 20 (Table 3). For variable X_{21}, which is Site 2 to Mill A, the VALUE is 10; and from X_{31}, Site 3 to Mill A, the VALUE is zero, so no loads will be hauled from Site 3 to Mill A. For X_{22}, Site 2 to Mill B, the daily number of truckloads will be 20, and so on.

Table 3.—XYZ Sawmill Company log truck haul schedule.

Logging site	Mill	Truckloads per day	Cost per load	Total cost
1	A	20	\$ 32	\$ 640
1	B	0	60	0
1	C	0	200	0
2	A	10	40	400
2	B	20	68	1,360
2	C	0	80	0
3	A	0	12	0
3	B	15	104	1,560
3	C	30	60	1,800
Total daily transportation costs for XYZ Sawmill Co.:				\$ 5,760

Transportation problem solution

Let's solve this problem using the transportation problem method, actually a simplified version of the simplex technique.

For this type of problem, all units available must be supplied. From the above problem, we see this in fact occurs: the sawmills use all 95 truckloads available. But isn't this unreasonable? How likely is it that supply always will equal demand? For practical purposes, it doesn't matter. As long as supply is adequate to meet demand, we can ignore the surplus and treat the total supply as equal to the total requirement.

Also, the number of constraints must equal the number of rows and number of columns when we set up our transportation problem. We have met this requirement also: three rows plus three columns equals six constraints, as shown above when we set up the problem to enter into the computer.

Another requirement is that the number of routes should equal the number of sources (sites), S, plus the number of destinations (mills), D, minus one, or

$$R = S + D - 1$$

For problems in which S + D – 1 does not equal the number of routes, either excess routes or *degeneracy* (more than one exiting cell) can occur. We will talk about these problems on page 14.

Let's set up our problem in a table format (Table 4). There are a number of methods of solving transportation problems. We will look at a common one, the *northwest corner method* (Table 5a). The northwest corner method is easy to use and requires only simple calculations. For other methods of solving transportation problems, refer to publications listed on page 35.

Table 4.—Hauling costs, log truckloads available, and log truckloads demanded.

Logging sites	Mill A	Mill B	Mill C	Available truckloads
Site 1	$ 32	$ 60	$ 200	20
Site 2	40	68	80	30
Site 3	120	104	60	45
Sawmill demand:	30	35	30	

As the method's name implies, we start work in the northwest corner, or the upper left cell, Site 1 Mill A. Make an allocation to this cell that will use either all the demand for that row or all the supply for that column, whichever is *smaller*. We see that Site 1's supply is smaller than Mill A's demand, so place a 20 in cell Site 1 Mill A. This eliminates row Site 1 from further consideration because we used all its supply. The next step is to move vertically to the next cell, Site 2 Mill A, and use the same criteria as before. Now the smaller value is column Mill A (versus 30 in row Site 2); there are only 10 truckloads demanded because 20 have been supplied by Site 1. So, move 10 into this cell. Now we can eliminate column Mill A from further consideration. Hint: when doing this with pencil and paper, it is convenient to draw a line across rows and down columns when they are no longer considered. This is especially useful when working on larger problems with more sources and destinations.

Table 5a.—XYZ Sawmill Company transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	32 20	60	200	20	
Site 2	40 10	68	80	30	
Site 3	120	104	60	45	
Demand	30	35	30	95	
v_j					

Move to the next cell, Site 2 Mill B (Table 5b). Move the smaller of the supply or demand values into this cell. Site 2 supply is 20 (remember, we used 10 of the 30 in Site 2 Mill A); Mill B demand is 35, so move 20 into this cell. We can eliminate row Site 2 from further consideration since all its supply (30) is used.

Move to cell Site 3 Mill B. The smaller of the margin values is 15. (Mill B demand is 15 because 20 of 35 truckloads have been supplied by Site 2.) Place 15 in this cell and eliminate column Mill B from further consideration since its demand has been met. Finally, move to cell Site 3 Mill C. Margin values are tied at 30 (v_{millC} = 30 and $u_{site\ 3}$ = 30 because we previously allocated 15 of the 45 available to Site 3 Mill B), so place 30 in this cell. We check to see whether we've met the requirement that the number of routes equals the number of sites plus the number of mills minus 1. In fact, we have met the requirement (3 + 3 – 1 = 5) and we have five routes.

Table 5b.—XYZ Sawmill Company transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	32 20	60	200	20	
Site 2	40 10	68 20	80	30	
Site 3	120	104 15	60 30	45	
Demand	30	35	30	95	
v_j					

Our initial basic feasible solution is:

$$32X_{11} + 40X_{21} + 68X_{22} + 104X_{32} + 60X_{33}$$

$$\$32(20) + \$40(10) + \$68(20) + \$104(15) + \$60(30) = \$5,760$$

However, this may not be our optimal solution. (Actually, we know it *is* optimal because of our LINDO solution, above.) Let's test to see whether the current tableau represents the optimal solution. We can do this because of duality theory (for a discussion of duality, see *Using Duality and Sensitivity Analysis to Interpret Linear Programming Solutions*, EM 8744, or another of the references on page 35). We can introduce two quantities, u_i and v_j, where u_i is the dual variable associated with row i and v_j is the dual variable associated with column j. From duality theory:

$$X_{ij} = u_i + v_j \text{ (Eq. 1)}$$

We can compute all u_i and v_j values from the initial tableau using Eq. 1.

$X_{11} = u_1 + v_2 = 32$
$X_{21} = u_2 + v_1 = 40$
$X_{22} = u_2 + v_2 = 68$
$X_{32} = u_3 + v_2 = 104$
$X_{33} = u_3 + v_3 = 60$

Since there are M + N unknowns and M + N – 1 equations, we can arbitrarily assign a value to one of the unknowns. A common method is to choose the row with the largest number of allocations (this is the number of cells where we have designated truckloads of logs). Row Site 2 and row Site 3 both have two allocations. We arbitrarily choose row Site 2 and set $u_2 = 0$. Using substitutions, we calculate:

$u_2 = 0$

$X_{21} = u_2 + v_1 = 40$	$0 + v_1 = 40$	$v_1 = 40$
$X_{22} = u_2 + v_2 = 68$	$0 + v_2 = 68$	$v_2 = 68$
$X_{11} = u_1 + v_1 = 32$	$u_1 + 40 = 32$	$u_1 = -8$
$X_{32} = u_3 + v_2 = 104$	$u_3 + 68 = 104$	$u_3 = 36$
$X_{33} = u_3 + v_3 = 60$	$36 + v_3 = 30$	$v_3 = -6$

Table 5c.—XYZ Sawmill Company transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	32 20	60	200	20	–8
Site 2	40 10	68 20	80	30	0
Site 3	120	104 15	60 30	45	36
Demand	30	35	30	95	
v_j	40	68	–6		

Arrange the u_i and v_j values around the table margin (Table 5c).

Here's how to recognize whether this tableau represents the optimal solution: for every nonbasic variable (those cells without any allocations), $X_{ij} - u_i - v_j \geq 0$ (Eq. 2). If Eq. 2 is true in every case, then the current tableau represents the optimal solution; if it is false in any one case, there is a better solution.

For cell Site 1 Mill B: 60 – (–8) – 68 ≥ 0 is true.
For cell Site 1 Mill C: 200 – (–8) – (–6) ≥ 0 is true.
For cell Site 2 Mill C: 80 – 0 – (–6) ≥ 0 is true.
For cell Site 3 Mill A: 120 – 36 – 40 ≥ 0 is true.

We know this represents the optimal solution and that $5,760 is our lowest cost to ship logs.

New XYZ Sawmill Company transportation problem

Let's examine another transportation problem. Suppose that the transportation schedule to the sawmills looks like that shown in Table 6a.

Demand equals supply: 45 truckloads. We will solve this problem using the northwest corner method. Start in Site 1 Mill A (northwest corner). Place a 5 in this cell (smaller of the truckloads demanded and those available). Since all the Mill A demand is satisfied, eliminate that row from further consideration (Table 6b).

Table 6a.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90	100	130	20	
Site 2	100	140	100	15	
Site 3	100	80	80	10	
Demand	5	20	20	45	
v_j					

Table 6b.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100	130	20	
Site 2	100	140	100	15	
Site 3	100	80	80	10	
Demand	5	20	20	45	
v_j					

Move to Site 1 Mill B (Table 6c). Place 15 in this cell. Remember, only 15 truckloads are available from Site 1 since 5 were used by Mill A, which is less than the 20 in Mill B demand margin. All 20 truckloads are used from Site 1, so we can eliminate this row from further consideration.

Table 6c.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	
Site 2	100	140	100	15	
Site 3	100	80	80	10	
Demand	5	20	20	45	
v_j					

Move to Site 2 Mill B (Table 6d). Place 5 (20 – 15 truckloads supplied by Site 1) in this cell. This eliminates column Mill B from further consideration.

Now we evaluate Site 2 Mill C. Place 10 in this cell and eliminate row Site 2 from further consideration (Table 6d).

Table 6d.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	
Site 2	100	140 5	100 10	15	
Site 3	100	80	80 10	10	
Demand	5	20	20	45	
v_j					

In cell Site 3 Mill C, place 10 (Table 6d). At this point, all constraints are satisfied.

Hauling costs are: \$90(5) + \$100(15) + \$140(5) + \$100(10) + \$80(10) = \$4,450

An analogy to the simplex method may help those familiar with that method of solving LP problems to better understand the transportation procedure.

Remember that when using simplex to solve LP minimization problems, we seek a *neighboring corner* (referring to a corner in dimensional space, not a corner in our transportation tables) to evaluate, choosing the one that improves costs the most. We reach this corner through a *pivot* procedure; that is, certain cells exchange status. A current nonempty cell (in the basis) becomes empty, or nonbasic, and an empty cell (currently not in the basis) becomes basic (or part of the solution). The new nonempty cell is called the *entering cell*, and the cell it replaces is called the *exiting cell* (just as in simplex). This will result in a less expensive shipping schedule. As with simplex, we continue this procedure until we reach a point where we can no longer improve; that is, we reach the most attractive corner and therefore the optimal solution.

Let's check to see whether the solution of $4,450 is optimal. We can check as we did with the preceding problem. Since there are M + N unknowns and M + N – 1 equations, we can arbitrarily assign a value to one of the unknowns and solve for the others. As above, choose the row that has the largest number of allocations (this is the number of cells where we have designated truckloads of logs). Site 1 and Site 2 have two allocations each, whereas Site 3 has only one allocation. We arbitrarily choose row Site 1 and set $u_1 = 0$.

Using substitutions, we calculate:

$u_1 + v_1 = 90$	$0 + v_1 = 90$	$v_1 = 90$
$u_1 + v_2 = 100$	$0 + v_2 = 100$	$v_2 = 100$
$u_2 + v_2 = 140$	$u_2 + 100 = 140$	$u_2 = 40$
$u_2 + v_3 = 100$	$40 + v_3 = 100$	$v_3 = 60$
$u_3 + v_3 = 80$	$u_3 + 60 = 80$	$u_3 = 20$

Arrange the u_i and v_j around the table as shown in Table 6e.

Check for the optimal solution using Eq. 2 ($X_{ij} – u_i – v_j \geq 0$) for each nonbasic cell. Remember, the nonbasic cells are those without an allocation. If the equation is true for each cell, then we have the optimal solution, and our lowest shipping cost is $4,450.

For cell Site 1 Mill C: $130 – 0 – 60 \geq 0$ is true.
For cell Site 2 Mill A: $100 – 40 – 90 \geq 0$ is false.

The second statement is false, so we know we can find a less costly shipping schedule. In order to continue, we need to determine the values in the remaining nonbasic cells. We are interested when the equation is false (when the answer is negative). Place –30 value in cell Site 2 Mill A (Table 6f, page 12).

Table 6e.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	0
Site 2	100	140 5	100 10	15	40
Site 3	100	80	80 10	10	20
Demand	5	20	20	45	
v_j	90	100	60		

For cell Site 3 Mill A: 100 – 20 – 90 > 0 is false. Place –10 in cell Site 3 Mill A (Table 6f).
For cell Site 3 Mill B: 80 – 20 – 100 > 0 is false. Place –40 in cell Site 3 Mill B (Table 6f).

Table 6f.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	0
Site 2	100 –30	140 5	100 10	15	40
Site 3	100 –10	80 –40	80 10	10	20
Demand	5	20	20	45	
v_j	90	100	60		

We will use what's known as the *closed loop path* for further iterations. Some rules will help us determine the closed loop path. One rule is there can be only one increasing and one decreasing cell in any row or column; and, except for the entering cell, all changes must involve nonempty (basic) cells.

For simplex (minimization problems), we choose the largest negative number as the new entering variable, and we do the same in this case. Site 3 Mill B has the largest negative value; therefore, we can reduce costs the most by allocating truckloads to this cell. We want to increase the value as much as possible, so place a "+" in cell Site 3 Mill B (Table 6g).

Table 6g.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	0
Site 2	100 –30	140 5 –	100 10 +	15	40
Site 3	100 –10	80 + –40	80 10 –	10	20
Demand	5	20	20	45	
v_j	90	100	60		

Since we want to increase this as much as possible, move to the next basic cell in the current solution, Site 2 Mill B. We need to keep everything in equilibrium, so place "–" in this cell (take 5 truckloads from here and allocate to Site 3 Mill B). Remember, we want to make a closed loop and keep supplies and demands balanced. Next, turn and go to the next basic cell in row Site 2 (Mill C) and place a "+" in this cell to indicate it must be increased by 5 to maintain equality in this row. Next, go to Site 3 Mill C and place a "–" in this cell. This completes a circuit. We have shifted the solution, and if we have completed the circuit correctly, equalities in our rows and columns have been maintained.

The following summarizes the changes.

Variable	Cell in closed loop	Before	Change	Now
X_{32}	Site 3 Mill B	Empty	+5	5
X_{22}	Site 2 Mill B	5	–5	Empty
X_{23}	Site 2 Mill C	10	+5	15
X_{33}	Site 3 Mill C	10	–5	5

This procedure must always make a complete circuit, starting and ending with the new entering variable. For any entering cell, the closed loop path is unique. It must step only into basic cells (all nonbasic cells are overstepped and ignored). At each cell where a "+" or "–" is entered, a *junction* is made; that is, turning either from vertical to horizontal or from horizontal to vertical. Sometimes this requires stepping over basic as well as nonbasic cells. If we reach a cell from which we cannot turn (because all other cells in its column or row are empty), then we must backtrack to the last turning point and go the other way or move forward to another nonempty cell in the same row or column.

Quantities will shift only in cells at the corners of the closed loop path. The amount shifted will equal the smallest quantity in the losing (–) cells. This is because each affected cell will change by plus or minus this value. If we used something greater than the smallest quantity, we would supply or demand quantities that are not available.

We now have the following solution.

$$\$90X_{11} + \$100X_{12} + \$100X_{23} + \$80X_{32} + \$80X_{33}$$
$$\$90(5) + \$100(15) + \$100(15) + \$80(5) + \$80(5) = \$4{,}250$$

We see that we did improve our costs: by $200, from $4,450 to $4,250. This new schedule is illustrated in Table 6h.

Calculate new margin values; that is, new u_i and v_j values. Look for the row with the most allocations. Sites 1 and 3 both have two allocations; row Site 2 has only one. We arbitrarily choose row Site 1 and set $u_1 = 0$ (Table 6h). Determine the other margin values through substitution.

Table 6h.—New XYZ Sawmill Company log transportation problem.

	Mill A	Mill B	Mill C	Supply	u_i
Site 1	90 5	100 15	130	20	0
Site 2	100	140	100 15	15	0
Site 3	100	80 5	80 5	10	–20
Demand	5	20	20	45	
v_j	90	100	100		

Some problems…
that have nothing to do with transporting goods are designed to be solved using the transportation method to ensure an integer answer.

$u_1 = 0$		
$u_1 + v_1 = 90$	$0 + v_1 = 90$	$v_1 = 90$
$u_1 + v_2 = 100$	$0 + v_2 = 100$	$v_2 = 100$
$u_3 + v_2 = 80$	$u_3 + 100 = 80$	$u_3 = -20$
$u_3 + v_3 = 80$	$-20 + v_3 = 80$	$v_3 = 100$
$u_2 + v_3 = 100$	$u_2 + 100 = 100$	$u_2 = 0$

We can now check to see whether our solution is optimal. Using Eq. 2, $X_{ij} - u_i - v_j \geq 0$, determine the values for the nonbasic cells (empty cells without any allocations):

For cell Site 1 Mill C: $130 - 0 - 100 \geq 0$ is true.
For cell Site 2 Mill A: $100 - 0 - 90 \geq 0$ is true.
For cell Site 2 Mill B: $140 - 0 - 100 \geq 0$ is true.
For cell Site 3 Mill A: $100 - (-20) - 90 \geq 0$ is true.

Because all statements are true, we know that $4,250 is the optimal solution and the lowest cost for our shipping schedule.

Integer solutions

When solving LP problems using simplex, an integer (whole number) solution is not guaranteed. We could end up with an optimal solution that makes fractions of things that really don't make much sense as fractions, such as producing 15.66 pickup trucks. But in transportation problems, if all the numbers representing the sources and the destinations are integers, then the answer also will be an integer. Because of this, some problems that have nothing to do with transporting goods are designed to be solved using the transportation method to ensure an integer answer.

Degeneracy in transportation problems

On page 6, we said if Sources + Destinations – 1 does not equal the number of routes, then either excess routes or degeneracy can occur when solving LP problems using the transportation method. When nonempty cells are fewer than necessary, it is impossible to obtain a unique set of row and column margin numbers (u_i and v_j) and, in some cases, to form closed-loop circuits. The program never converges on an optimal solution but continues to cycle through the same solutions over and over again.

Consider the following problem. XYZ Sawmill Company purchased another sawmill and formulated a new logging schedule (Table 7a). We will use the northwest corner rule to get the initial solution to the tableau.

Again, we put the cost to ship a truckload of logs from a site to a mill in the upper right corner. Looking at supply and demand, we place the smallest value, 10, in the northwest corner cell. Row Site 1 now can be eliminated from further consideration.

Table 7a.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19 10	7	3	21	10	
Site 2	15	21	18	6	30	
Site 3	11	14	15	22	20	
Demand	15	10	20	15	60	
v_j						

Move down to Site 2 Mill A and again assign the smallest value. This is 5 (from the demand margin, because 10 have been assigned already), and we place that value in cell Site 2 Mill A (Table 7b). Column Mill A can now be eliminated from further consideration.

Move to Site 2 Mill B and place the smallest of the margin values, 10, in that cell (Table 7b). Mill B column can now be eliminated from further consideration. Continuing to work in Table 7b, move to Site 2 Mill C cell and assign the smallest margin value to this cell. In this case, we will assign only 15 of the 20 to keep the equality of the row supply. Move to Site 3 Mill C and assign the remaining 5 from the demand margin to this cell. Finally, move to the Site 3 Mill D cell and assign 15 from the demand margin to this cell. If we have assigned all our values correctly, then all row and column equalities will have been preserved. In other words, all constraints will have been satisfied.

Table 7b.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19 10	7	3	21	10	
Site 2	15 5	21 10	18 15	6	30	
Site 3	11	14	15 5	22 15	20	
Demand	15	10	20	15	60	
v_j						

Our initial feasible solution is:

Daily shipping costs = $19(10) + $15(5) + $21(10) + $18(15) + $15(5) + $22(15) = $1,150

We check to see whether this is an optimal solution. Start by calculating u_i and v_j values. Row Site 2 has the most allocations, so assign u_2 a value of zero and solve for the remainder of the margin values:

$u_2 = 0$

$u_2 + v_1 = 15$	$0 + v_1 = 15$	$v_1 = 15$
$u_1 + v_1 = 19$	$u_1 + 15 = 19$	$u_1 = 4$
$u_2 + v_2 = 21$	$0 + v_2 = 21$	$v_2 = 21$
$u_2 + v_3 = 18$	$0 + v_3 = 18$	$v_3 = 18$
$u_3 + v_3 = 15$	$u_3 + 18 = 15$	$u_3 = -3$
$u_3 + v_4 = 22$	$-3 + v_4 = 22$	$v_4 = 25$

Place the margin values into Table 7c. Use Eq. 2 to determine whether this is the optimal solution.

For Cell Site 1 Mill B: $7 - 4 - 21 \geq 0$ is false. Therefore, we know this is not the optimal solution. Place –18 in Site 1 Mill B (Table 7c).
For Cell Site 1 Mill C: $3 - 4 - 18 \geq 0$ is false. Place –19 in Site 1 Mill C (Table 7c).
For Cell Site 1 Mill D: $21 - 4 - 25 \geq 0$ is false. Place –8 in Site 1 Mill D (Table 7c).
For Cell Site 2 Mill D: $6 - 0 - 25 \geq 0$ is false. Place –19 in Site 2 Mill D (Table 7c).
For Cell Site 3 Mill A: $11 - (-3) - 15 \geq 0$ is false. Place –1 in Site 3 Mill A (Table 7c).
For Cell Site 3 Mill B: $14 - (-3) - 21 \geq 0$ is false. Place –4 in Site 3 Mill B (Table 7c).

Table 7c.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19 10	7 –18	3 –19	21 –8	10	4
Site 2	15 5	21 10	18 15	6 –19	30	0
Site 3	11 –1	14 –4	15 5	22 15	20	–3
Demand	15	10	20	15	60	
v_j	15	21	18	25		

Remember, just as for the simplex LP method, we choose the largest negative number as the new entering variable. Site 1 Mill C and Site 2 Mill D tie for the largest negative value. We can reduce costs the most by allocating truckloads to one of these cells.

As with most LP ties, we can choose either one. We arbitrarily choose Site 1 Mill C. We want to increase the value as much as possible, so put a "+" in Site 1 Mill C (Table 7d). To keep the supply equality for Site 1, place a "–" in Site 1 Mill A. Continue along the closed loop circuit, making sure not to violate supply and demand equalities. Place a "+" in Site 2 Mill A and, finally, a "–" in Site 2 Mill C (Table 7d).

Table 7d.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19 10 –	7 –18	3 + –19	21 –8	10	4
Site 2	15 5+	21 10	18 15–	6 –19	30	0
Site 3	11 –1	14 –4	15 5	22 15	20	–3
Demand	15	10	20	15	60	
v_j	15	21	18	25		

Now we can change the trucking schedule. Remember that changes occur only at the corners. Summarizing the changes:

Cell in closed loop	Before	Change	Now
Site 1 Mill C	Empty	+10	10
Site 1 Mill A	10	–10	Empty
Site 2 Mill A	5	+10	15
Site 2 Mill C	15	–10	5

Table 7e shows the trucking schedule change. Site 1 Mill A becomes empty and therefore is the exiting cell. Next, we calculate our new trucking costs: \$3(10) + \$15(15) + \$21(10) + \$18(5) + \$15(5) + \$22(15) = \$960.

We improved our shipping costs by \$190, from \$1,150 to \$960. But, is this the optimal solution? We can check by calculating the u_i and v_j values and then using Eq. 2.

Table 7e.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	
Site 2	15 15	21 10	18 5	6	30	
Site 3	11	14	15 5	22 15	20	
Demand	15	10	20	15	60	
v_j						

Site 2 row has the most allocations, so set $u_2 = 0$ and solve for the other values:

$u_2 = 0$		
$u_2 + v_1 = 15$	$0 + v_1 = 15$	$v_1 = 15$
$u_2 + v_2 = 21$	$0 + v_2 = 21$	$v_2 = 21$
$u_2 + v_3 = 18$	$0 + v_3 = 18$	$v_3 = 18$
$u_1 + v_3 = 3$	$u_1 + 18 = 3$	$u_1 = -15$
$u_3 + v_3 = 15$	$u_3 + 18 = 15$	$u_3 = -3$
$u_3 + v_4 = 22$	$-3 + v_4 = 22$	$v_4 = 25$

Table 7f.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	–15
Site 2	15 15	21 10	18 5 –	6 + –19	30	0
Site 3	11 –1	14 –4	15 5 +	22 15 –	20	–3
Demand	15	10	20	15	60	
v_j	15	21	18	25		

Place these values in Table 7f and use Eq. 2 to check for an optimal solution. For Eq. 2, use the nonbasic cells (those with no allocation). If all statements are true, we have the optimal solution; that is, the lowest cost trucking schedule.

For cell Site 1 Mill A: $19 - (-15) - 15 \geq 0$ is true.
For cell Site 1 Mill B: $7 - (-15) - 21 \geq 0$ is true.
For cell Site 1 Mill D: $21 - (-15) - 25 \geq 0$ is true.
For cell Site 2 Mill D: $6 - 0 - 25 \geq 0$ is false. We know this is not the optimal solution. Place –19 in Site 2 Mill D (Table 7f).
For cell Site 3 Mill A: $11 - (-3) - 15 \geq 0$ is false. Place –1 in Site 3 Mill A (Table 7f).
For cell Site 3 Mill B: $14 - (-3) - 21 \geq 0$ is false. Place –4 in Site 3 Mill B (Table 7f).

Continuing to work with Table 7f, we see that Site 2 Mill D is our new entering variable (that with the largest negative value and therefore the one that can reduce costs the most). Place a "+" in this cell. Complete the closed loop by placing a "–" in Site 2 Mill C, a "+" in Site 3 Mill C, and a "–" in Site 3 Mill D.

Summarizing the changes:

Cell in closed loop	Before	Change	Now
Site 2 Mill D	Empty	+5	5
Site 2 Mill C	5	–5	Empty
Site 3 Mill C	5	+5	10
Site 3 Mill D	15	–5	10

The new schedule is in Table 7g. The new solution is: \$3(10) + \$15(15) + \$21(10) + \$6(5) + \$15(10) + \$22(10) = \$865. We improved our shipping costs by \$95, from \$960 to \$865. Check to see whether this is the optimal solution. Row Site 2 has the most allocations, so set $u_2 = 0$. Solve for the other values and place into Table 7g.

Table 7g.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7 –18	3 10	21	10	4
Site 2	15 15	21 10 –	18 –1	6 5 +	30	0
Site 3	11 –20	14 + –23	15 10	22 10 –	20	16
Demand	15	10	20	15	60	
v_j	15	21	–1	6		

$u_2 = 0$

$u_2 + v_1 = 15$	$0 + v_1 = 15$	$v_1 = 15$
$u_2 + v_2 = 21$	$0 + v_2 = 21$	$v_2 = 21$
$u_2 + v_4 = 6$	$0 + v_4 = 6$	$v_4 = 6$
$u_3 + v_4 = 22$	$u_3 + 6 = 22$	$u_3 = 16$
$u_3 + v_3 = 15$	$16 + v_3 = 15$	$v_3 = -1$
$u_1 + v_3 = 3$	$u_1 + (-1) = 3$	$u_1 = 4$

Use Eq. 2 to check whether this is the optimal solution:

For cell Site 1 Mill A: $19 - 4 - 15 \geq 0$ is true.
For cell Site 1 Mill B: $7 - 4 - 21 \geq 0$ is false. Place –18 in Site 1 Mill B (Table 7g).
For cell Site 1 Mill D: $21 - 4 - 6 \geq 0$ is true.
For cell Site 2 Mill C: $18 - 0 - (-1) \geq 0$ is true.
For cell Site 3 Mill A: $11 - 16 - 15 \geq 0$ is false. Place –20 in Site 3 Mill A (Table 7g).
For cell Site 3 Mill B: $14 - 16 - 21 \geq 0$ is false. Place –23 in Site 3 Mill B (Table 7g).

Choose the largest negative value (this will reduce costs the most) as the new entering variable. Place a "+" in Site 3 Mill B. Move to Site 2 Mill B, place a "–" in it, then place a "+" in Site 2 Mill D (we have to jump over the nonbasic cell), and then close the loop by placing a "–" in Site 3 Mill D (we ignore Site 2 Mill C and Site 3 Mill C).

Summarizing the changes in the trucking schedule:

Cell in closed loop	Before	Change	Now
Site 3 Mill B	Empty	+10	10
Site 2 Mill B	10	–10	Empty
Site 2 Mill D	5	+10	15
Site 3 Mill D	10	–10	Empty

Remember that the empty cell becomes the exiting cell. For this iteration, though, we have a tie. Both Site 2 Mill B and Site 3 Mill D cells have become empty. Both cells lose all their allocation and are reduced to zero. However, now we have two routes fewer than the number of rows and columns. When there are fewer than S + D – 1 routes, sometimes it is not possible to form closed loops, and we cannot solve for a unique set of row and column margin numbers (u_i and v_j). Too few routes also can lead to degeneracy. To avoid degeneracy, we need to treat one of the exiting cells as a nonempty or basic cell. Again, because ties almost always can be arbitrarily broken in LP problems, we can choose either one as the exiting cell and treat the other as a basic cell. We will choose Site 2 Mill B as the exiting cell and place a zero in Site 3 Mill D cell. Our new shipping schedule is reflected in Table 7h.

Our new shipping cost is: \$3(10) + \$15(15) + \$6(15) + \$14(10) + \$15(10) + \$22(0) = \$635. Our cost has been reduced from \$865 to \$635 for an improvement of \$230.

Table 7h.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	–12
Site 2	15 15	21	18	6 15	30	–16
Site 3	11	14 10	15 10	22 0	20	0
Demand	15	10	20	15	60	
v_j	17	14	15	22		

Site 3 row has three allocations, so we can set $u_3 = 0$ and solve for the other margin values (Table 7h):

$u_3 = 0$

$u_3 + v_2 = 14$	$0 + v_2 = 14$	$v_2 = 14$
$u_3 + v_3 = 15$	$0 + v_3 = 15$	$v_3 = 15$
$u_3 + v_4 = 22$	$0 + v_4 = 22$	$v_4 = 22$
$u_2 + v_4 = 6$	$u_2 + 22 = 6$	$u_2 = -16$
$u_2 + v_1 = 15$	$-16 + v_1 = 15$	$v_1 = 17$
$u_1 + v_3 = 3$	$u_1 + 15 = 3$	$u_1 = -12$

Use Eq. 2 to check to see whether this is the optimal solution.
For cell Site 1 Mill A: $19 - (-12) - 17 \geq 0$ is true.
For cell Site 1 Mill B: $7 - (-12) - 14 \geq 0$ is true.
For cell Site 1 Mill D: $21 - (-12) - 22 \geq 0$ is true.
For cell Site 2 Mill B: $21 - (-16) - 14 \geq 0$ is true.
For cell Site 2 Mill C: $18 - (-16) - 15 \geq 0$ is true.
For cell Site 3 Mill A: $11 - 0 - 17 \geq 0$ is false. Place –6 in Site 3 Mill A (Table 7i).

The new entering cell is Site 3 Mill A, and we place a "+" in this cell. Increasing the allocation to this cell will save $6 per truckload. To finish the closed loop, place a "–" in Site 2 Mill A, a "+" in Site 2 Mill D, and a "–" in Site 3 Mill D (Table 7i).

Table 7i.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	
Site 2	15 15 –	21	18	6 15 +	30	
Site 3	11 + –6	14 10	15 10	22 0 –	20	
Demand	15	10	20	15	60	
v_j						

We have a problem because one of the cells to be reduced—that is, one with a minus sign in it—is the zero cell, Site 3 Mill D. When this occurs, we can shift the zero to a cell that is to be increased rather than reduced. As long as we keep the zero in the same row or column, we do not change the shipping schedule or violate a constraint. We know we want to increase the allocation to Site 3 Mill A cell. Shifting the zero to the Site 3 Mill A cell gives us the new shipping schedule in Table 7j (page 22).

Table 7j.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	
Site 2	15 15 –	21	18 +	6 15	30	
Site 3	11 0 +	14 10	15 10 –	22	20	
Demand	15	10	20	15	60	
v_j						

The cost for this schedule is exactly the same, but we no longer need to worry about reducing a cell that is already zero. Make the changes by moving 10 from Site 3 Mill C to Site 2 Mill C and moving 10 from Site 2 Mill A to Site 3 Mill A. Summarizing the changes:

Cell in closed loop	Before	Change	Now
Site 3 Mill C	10	–10	Empty
Site 2 Mill C	Empty	+10	10
Site 2 Mill A	15	–10	5
Site 3 Mill A	0	+10	10

Table 7k.—XYZ Sawmill Company's three harvesting sites and four sawmills.

	Mill A	Mill B	Mill C	Mill D	Supply	u_i
Site 1	19	7	3 10	21	10	–15
Site 2	15 5	21	18 10	6 15	30	0
Site 3	11 10	14 10	15	22	20	–4
Demand	15	10	20	15	60	
v_j	15	18	18	6		

Note that the solution is no longer degenerate (more than one exiting cell). The new shipping schedule is in Table 7k.

The new shipping schedule is: \$3(10) + \$15(5) + \$18(10) + \$6(15) + \$11(10) + \$14(10) = \$625. We reduced our shipping costs from \$635 to \$625 for a savings of \$10. But, is this the optimal solution?

Calculate our margin values by setting $u_2 = 0$ and solving for the other values:

$u_2 = 0$

$u_2 + v_1 = 15$	$0 + v_1 = 15$	$v_1 = 15$
$u_2 + v_3 = 18$	$0 + v_3 = 18$	$v_3 = 18$
$u_2 + v_4 = 6$	$0 + v_4 = 6$	$v_4 = 6$
$u_3 + v_1 = 11$	$u_3 + 15 = 11$	$u_3 = -4$
$u_3 + v_2 = 14$	$-4 + v_2 = 14$	$v_2 = 18$
$u_1 + v_3 = 3$	$u_1 + 18 = 3$	$u_1 = -15$

Use Eq. 2 to see whether this is the optimal solution.

For cell Site 1 Mill A: 19 – (–15) – 15 ≥ 0 is true.
For cell Site 1 Mill B: 7 – (–15) – 18 ≥ 0 is true.
For cell Site 1 Mill D: 21 – (–15) – 6 ≥ 0 is true.
For cell Site 2 Mill B: 21 – 0 – 18 ≥ 0 is true.
For cell Site 3 Mill C: 15 – (–4) – 18 ≥ 0 is true.
For cell Site 3 Mill D: 22 – (–4) – 6 ≥ 0 is true.

All statements are true, so we cannot improve the solution further. Our optimal shipping cost for this schedule is $625.

Assignment problem and Vogel's Approximation

As stated earlier, problems other than transportation problems can be solved using the transportation method. To illustrate, we will solve an assignment problem using the transportation method. For this example, we will use another starting procedure called *Vogel's Approximation.* Vogel's Approximation is somewhat more complex than the northwest corner method, but it takes cost information into account and often gives an initial solution closer to the optimal solution. You should be able to use either the northwest corner method or Vogel's Approximation to solve transportation or assignment problems.

Example: Job-shop assignment problem

Six individuals are to work on six different machines at the ACME Furniture Company. The machines shape, surface, and drill to make a finished wood component. The work will be sequential so that a finished component is produced at the end of the manufacturing process. All individuals are skilled on all machines, but each worker is not equally skilled on each machine. Table 8 lists the time it takes a worker to complete a task at a machine center.

Table 8.—Average number of seconds required to complete a task.

	Machine center					
Individual	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill
Worker 1	13	22	19	21	16	20
Worker 2	18	17	24	18	22	27
Worker 3	20	22	23	24	17	31
Worker 4	14	19	13	30	23	22
Worker 5	21	14	17	25	15	23
Worker 6	17	23	18	20	16	24

The average time to complete a task takes the place of unit shipping costs in the transportation examples above. Instead of reducing transportation costs, we are reducing the time to process a piece of wood into a finished component. We can set up our first table, Table 9a.

Table 9a.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	Available workers
Worker 1	13	22	19	21	16	20	1
Worker 2	18	17	24	18	22	27	1
Worker 3	20	22	23	24	17	31	1
Worker 4	14	19	13	30	23	22	1
Worker 5	21	14	17	25	15	23	1
Worker 6	17	23	18	20	16	24	1
# of Workers	1	1	1	1	1	1	6

Margin values are the number of workers available and the number of workers required to run a machine, one in each case. The number of workers available equals the number of workers required: six. This meets the criterion we established for the transportation examples, where supply had to at least equal demand.

The objective is to minimize the total amount of time (labor) it takes to process a finished wood component. What about the cell values? They are the average time a worker spends at a machine center, represented as:

X_{ij} = amount of time worker i (i = 1 – 6) is assigned to machine center j (j = 1 – 6), as shown in Table 9b.

Let C_{ij} = average time to complete a task in row i and column j.

Table 9b.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 X_{11}	22 X_{12}	19 X_{13}	21 X_{14}	16 X_{15}	20 X_{16}	1	*3*
Worker 2	18 X_{21}	17 X_{22}	24 X_{23}	18 X_{24}	22 X_{25}	27 X_{26}	1	*1*
Worker 3	20 X_{31}	22 X_{32}	23 X_{33}	24 X_{34}	17 X_{35}	31 X_{36}	1	*3*
Worker 4	14 X_{41}	19 X_{42}	13 X_{43}	30 X_{44}	23 X_{45}	22 X_{46}	1	*1*
Worker 5	21 X_{51}	14 X_{52}	17 X_{53}	25 X_{54}	15 X_{55}	23 X_{56}	1	*1*
Worker 6	17 X_{61}	23 X_{62}	18 X_{63}	20 X_{64}	16 X_{65}	24 X_{66}	1	*1*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*2*		

*Available workers
**Row differences between lowest C_{ij}'s
***Column differences between lowest C_{ij}'s

First, use Vogel's Approximation to find a starting cell. Find the difference between the two lowest costs (shortest times) in each row and column. (In this case, time to complete a task is treated the same as shipping costs.) For row 1, the two shortest times are for Surfacer, 13 seconds, and Sander 2, 16 seconds. The difference (16 – 13) is 3. For column 1, the two shortest times are for Worker 1, 13 seconds, and Worker 4, 14 seconds. The difference (14 – 13) is 1. Find the difference for the two shortest times for all other rows and columns and place in Table 9b (shown in italics, as RD and CD).

Look for the largest value: 4, in column 3. Make the maximum assignment (in this case, 1) to the cheapest (shortest time) cell in column 3; that is, put a 1 in cell X_{43} (Table 9c). We can cross out row 4 (Worker 4) and column 3 (Sander) since we've met the requirements for that row and that column. New differences are calculated using the cells that remain (Table 9d). The greatest difference, 4, is in

Table 9c.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17	31	1	*3*
Worker 4	14	19	13 1	30	23	22	1	*1*
Worker 5	21	14	17	25	15	23	1	*1*
Worker 6	17	23	18	20	16	24	1	*1*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*2*		

Table 9d.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17	31	1	*3*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14	17	25	15	23	1	*1*
Worker 6	17	23	18	20	16	24	1	*1*
# of Workers	1	1	1	1	1	1	6	
CD***	*4*	*3*	*4*	*2*	*1*	*3*		

*Available workers
**Row differences between lowest C_{ij}'s
***Column differences between lowest C_{ij}'s

column 1. The cheapest unshaded cell in column 1 is X_{11}. We assign the maximum allowed allocation, 1, to this cell (Table 9e). Since neither row 1 nor column 1 can receive more allocations, cross them out (Table 9e).

Calculate new differences for remaining unshaded rows and columns (Table 9f).

The greatest difference is 5 from row 3. The cheapest unshaded cell is X_{35} with a value of 17. Assign the maximum allocation, 1, to this cell and cross out row 3 and column 5 (Table 9g).

Table 9e.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17	31	1	*5*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14	17	25	15	23	1	*1*
Worker 6	17	23	18	20	16	24	1	*2*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*1*		

Table 9f.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17	31	1	*5*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14	17	25	15	23	1	*1*
Worker 6	17	23	18	20	16	24	1	*4*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*1*		

Table 9g.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17 1	31	1	*5*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14	17	25	15	23	1	*1*
Worker 6	17	23	18	20	16	24	1	*4*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*1*		

Calculate new differences of unshaded cells for rows and columns. Find the greatest difference and allocate the maximum, 1, to the cheapest cell (Table 9h). The greatest difference is 9 in row 5, and the cheapest unshaded cell is X_{52} with a value of 14.

Continuing to work with Table 9h, allocate the maximum of 1 to this cell and cross out row 5 and column 2. Calculate differences for rows and columns. The greatest difference is 9 for row 2. The cheapest unshaded cell is X_{24} with a value of 18. Assign 1 to this cell and block out row 2 and column 4 (Table 9i). There is only one unshaded cell remaining. We can assign 1 to cell X_{66} (Table 9i).

Table 9h.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*1*
Worker 3	20	22	23	24	17 1	31	1	*5*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14	17	25	15	23	1	*9*
Worker 6	17	23	18	20	16	24	1	*3*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*1*		

Table 9i.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	*3*
Worker 2	18	17	24	18	22	27	1	*9*
Worker 3	20	22	23	24	17 1	31	1	*5*
Worker 4	14	19	13 1	30	23	22	1	*6*
Worker 5	21	14 1	17	25	15	23	1	*9*
Worker 6	17	23	18	20	16	24 1	1	*4*
# of Workers	1	1	1	1	1	1	6	
CD***	*1*	*3*	*4*	*2*	*1*	*3*		

*Available workers
**Row differences between lowest C_{ij}'s
***Column differences between lowest C_{ij}'s

Table 9j.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	RD**
Worker 1	13 1	22	19	21	16	20	1	
Worker 2	18	17	24	18 1	22	27	1	
Worker 3	20	22	23	24	17 1	31	1	
Worker 4	14	19	13 1	30	23	22	1	
Worker 5	21	14 1	17	25	15	23	1	
Worker 6	17	23	18	20	16	24 1	1	
# of Workers	1	1	1	1	1	1	6	
CD***								

*Available workers
**Row differences between lowest C_{ij}'s
***Column differences between lowest C_{ij}'s

Table 9j shows the assignment schedule using Vogel's Approximation.

Unlike the northwest corner method, Vogel's Approximation does not always result in the required number of nonempty cells. Remember that in order to use the transportation method to solve LP problems, the number of routes (or, in this case, assignments) must equal the number of sources or sites (in this case, workers) plus the number of destinations (in this case, jobs) minus one; that is, Assignments = Workers + Jobs – 1; that is, Assignments = 6 + 6 – 1. We can see that for our problem, we have six nonempty cells but need eleven. We need to assign zeroes to five empty cells.

Using the transportation method, find the marginal row and column values. In the preceding transportation problems, we assigned a row value of zero to the row that had the most shipping routes (analogous to our job assignments). Since all rows have the same allocation of 1, we will arbitrarily assign a margin value of zero to row 1 (Table 9k).

Table 9k—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	u_i
Worker 1	13 1	22	19	21	16	20	1	0
Worker 2	18	17 0	24	18 1	22	27	1	8
Worker 3	20	22	23	24	17 1	31 0	1	17
Worker 4	14 0	19	13 1	30	23	22	1	1
Worker 5	21	14 1	17 0	25	15	23	1	5
Worker 6	17	23	18	20 0	16	24 1	1	10
# of Workers	1	1	1	1	1	1	6	
v_j	13	9	12	10	0	14		

We are able to calculate v_1 = 13, 0 + 13 = 13. To continue, we need another nonempty cell (Table 9k). We check for the cheapest (least time) nonempty cell and place a zero in it. This would be cell X_{41}, with a value of 14 seconds (Table 9k). We can now calculate u_4 as 1.

Next we see that we can calculate v_3 as 1 + 12 = 13. Again, we have reached a point where no further calculations can be made. Look at column 2 and place a zero in the cheapest empty cell, X_{22}. Calculate the margin value for row 2: 8 + 9 = 17. Next, calculate the margin value for column 4 as 8 + 10 = 18. Calculate the margin value for row 6 as 10 + 10 = 20. Next, calculate the margin value for column 6 as 10 + 14 = 24.

Again, we have reached an impasse. We need to find margin values for row 3 and column 5. If we place a zero in cell X_{36} we will be able to find our two remaining margin values. Calculate the margin value for row 3 as 17 + 14 = 31, and the margin value for column 5 as 17 + 0 = 17 (Table 9k).

Now, we have the required Assignments = Workers + Jobs – 1.

Use the margin values to determine the empty cell differences and place these in the left bottom corner of each empty cell (Table 9l). For example, cell X_{12}, 0 + 9 + value = 22. Therefore the value = 13. Fill in the remainder of the empty cell differences (Table 9l).

Table 9l—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	u_i
Worker 1	13 1	22 13	19 7	21 11	16 16	20 6	1	0
Worker 2	18 –3	17 0	24 4	18 1	22 14	27 5	1	8
Worker 3	20 –10	22 –4	23 –6	24 –3	17 1	31 0	1	17
Worker 4	14 0	19 9	13 1	30 19	23 22	22 7	1	1
Worker 5	21 3	14 1	17 0	25 10	15 10	23 4	1	5
Worker 6	17 –6	23 4	18 –4	20 0	16 6	24 1	1	10
# of Workers	1	1	1	1	1	1	6	
v_j	13	9	12	10	0	14		

*Available workers

The most negative cell, X_{31} with a value of –10, is the new entering cell. Use the closed-loop path to determine the exiting cell (Table 9m). Lines with arrows highlight the closed-loop path. Note that all the losing cells contain zeroes. Therefore, the maximum quantity to be reallocated along the path is zero, and one of the losing cells will go blank in the next solution. Break the tie by choosing the most expensive losing cell, X_{31}, and move the zero from cell X_{36} to cell X_{31}. Since the value is zero, however, the total time savings is 10 x 0 = 0.

We will leave it up to you to go through the next few iterations. They involve only moving zeroes; therefore, this assignment is optimal and leads to the fastest processing time.

The processing time is calculated as:

Worker 1	Surfacer	13 sec
Worker 2	Router	18 sec
Worker 3	Sander 2	17 sec
Worker 4	Sander 1	13 sec
Worker 5	Lathe	14 sec
Worker 6	Drill	24 sec
Total processing time/component		99 sec

There is an alternative optimal solution to this assignment problem, but we leave it to you to discover.

Table 9m.—Job-shop assignment problem for ACME Furniture Company.

	Surfacer	Lathe	Sander 1	Router	Sander 2	Drill	AW*	u_i
Worker 1	13 1	22 13	19 7	21 11	16 16	20 6	1	0
Worker 2	18 –3	17 0 –	24 4	18 1 +	22 14	27 5	1	8
Worker 3	20 + –10	22 –4	23 –6	24 –3	17 1	31 0 –	1	17
Worker 4	14 0 –	19 9	13 1 +	30 19	23 22	22 7	1	1
Worker 5	21 3	14 1 +	17 0 –	25 10	15 10	23 4	1	5
Worker 6	17 –6	23 4	18 –4	20 0 –	16 6	24 1 +	1	10
# of Workers	1	1	1	1	1	1	6	
v_j	13	9	12	10	0	14		

Conclusion

Transportation problems can be solved using the simplex method; however, the simplex method involves time-consuming computations. And, it is easy to make a mistake when working the problems by hand.

An advantage to the transportation method is that the solution process involves only the main variables; artificial variables are not required, as they are in the simplex process. In fact, after applying the northwest corner rule, the problem is as far along as it would be using simplex after eliminating the artificial variables.

Simplex requires a minimum of iterations (each represented by another simplex tableau) equal to the number of rows plus columns minus 1. This is the minimum; many problems require more iterations. Calculations are much easier to obtain with a new transportation table than a new simplex tableau. After practice, a relatively large transportation problem can be solved by hand. This is not true when solving large LP problems using the simplex method.

Finally, some LP computer programs are set up to solve both simplex and transportation problems. When the problems are introduced as transportation problems, the computer algorithms can solve them much faster.

Although developed to solve problems in transporting goods from one location to another, the transportation method can be used to solve other problems—such as the assignment example above—as long as the problem can be set up in the transportation problem form.

To learn more about transportation problems, check out the references below.

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